

# Quantum tricriticality in transverse Ising-like systems

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the date of receipt and acceptance should be inserted later

**Abstract.** The quantum tricriticality of  $d$ -dimensional transverse Ising-like systems is studied by means of a perturbative renormalization group approach focusing on static susceptibility. This allows us to obtain the phase diagram for  $3 \leq d < 4$ , with a clear location of the critical lines ending in the conventional quantum critical points and in the quantum tricritical one, and of the tricritical line for temperature  $T \geq 0$ . We determine also the critical and the tricritical shift exponents close to the corresponding ground state instabilities. Remarkably, we find a tricritical shift exponent identical to that found in the conventional quantum criticality and, by approaching the quantum tricritical point increasing the non-thermal control parameter  $r$ , a crossover of the quantum critical shift exponents from the conventional value  $\phi = 1/(d-1)$  to the new one  $\phi = 1/2(d-1)$ . Besides, the projection in the  $(r, T)$ -plane of the phase boundary ending in the quantum tricritical point and crossovers in the quantum tricritical region appear quite similar to those found close to an usual quantum critical point. Another feature of experimental interest is that the amplitude of the Wilsonian classical critical region around this peculiar critical line is sensibly smaller than that expected in the quantum critical scenario. This suggests that the quantum tricriticality is essentially governed by mean-field critical exponents, renormalized by the shift exponent  $\phi = 1/2(d-1)$  in the quantum tricritical region.

**PACS.** 64.60.ae Renormalization-group theory – 64.70.Tg Quantum phase transitions – 64.60.Kw Multicritical points – 75.40.Cx Static properties (order parameter, static susceptibility, heat capacities, critical exponents, etc.)

## 1 Introduction

The last decades have seen a renovated interest in the topic of quantum phase transitions (QPTs), which have had a surprising impact in explaining several exotic low-temperature properties of a variety of innovative materials [1,2,3,4,5] which do not fit conventional many-body theories. Moreover, even though a QPT is strictly speaking a zero-temperature instability, its experimental manifestations are clearly observed at finite temperature, within a rather wide region around the quantum critical point (QCP) [4,5].

Recent experimental [6,7,8,9] and theoretical [10,11] analysis of ferromagnetic QPTs have shown that the second-order phase transition turns into a first-order one when approaching the absolute zero temperature. Thus, the emerging scenario indicates the presence of a tricritical point in proximity of a QPT, that prevents the long-range order to be continuously connected with the disordered phase when the quantum fluctuations become dominant.

In principle, it is possible to tune opportunely the control parameters in order to move the tricritical point to zero temperature, creating then a quantum tricritical point (QTCP). The previous considerations pose the prob-

lem on a broader context and point the interest to the nature of the critical fluctuations around a QTCP, as well as about their possible physical implications.

Indeed, there are experimental evidences for some compounds suggesting that they are close to a QTCP. Examples are given by the oxides  $\text{Sr}_3\text{Ru}_2\text{O}_7$  [12,13,14] and  $\text{Pr}_2\text{CuO}_4$  [15], the heavy fermions  $\text{URhGe}$  [16] and  $\text{YbRh}_2\text{Si}_2$  [17,18], and the pnictide superconductor  $\text{LaFeAsO}$  [19]. Let us present some more detail for some of the mentioned systems. Concerning the ruthenate compound  $\text{Sr}_3\text{Ru}_2\text{O}_7$ , a metamagnetic QCP was found [12], where the field angle  $\theta$  (measured with respect to the  $ab$  plane) acts as a tuning parameter, allowing the construction of a  $(H, \theta, T)$  phase diagram, where  $H$  is the magnetic field and  $T$  is the temperature, for metamagnetism and quantum criticality [13]. The metamagnetic transition tracks a first-order line in the  $(H, \theta)$ -plane which terminates at a QCP when the temperature, and thus the energy, scale associated with the metamagnetic transition is tuned to zero. Successive studies [14] on ultraclean samples of  $\text{Sr}_3\text{Ru}_2\text{O}_7$  have revealed striking anomalies in magnetism and transport in the vicinity of the QCP. In particular, measurements of resistivity and ac magnetic susceptibility have shown an in-

tricate phase diagram where the line of first-order metamagnetic transition in the  $(H, \theta)$  plane appears to bifurcate in two transition lines (observed for  $H \parallel c$ ). Such a complex phase behavior can be ascribed to a “symmetry-broken” tricritical point structure [14]. Thus the bifurcation mechanism seems to provide an opportunity to realize quantum tricritical behavior in  $\text{Sr}_3\text{Ru}_2\text{O}_7$ . This feature seems to happen also in the heavy-fermion compound  $\text{URhGe}$  [20]. This material has been intensively studied due to the emergence of superconductivity close to its ferromagnetic QCP. Detailed experimental analysis of the magnetic field vs. temperature phase diagram show, at very low temperatures, the presence of a tricritical point at which the continuous transition line bifurcates [16]. The superconducting phase in  $\text{URhGe}$  occurs around this bifurcation close to zero temperature. This situation suggests that  $\text{URhGe}$  is close to a QTCP and not just an ordinary QCP [16], but it has not so far been examined theoretically. Studies in this sense could give new interesting information about the mechanism driving the unconventional superconductivity. Finally, very recently it has been speculated that also iron pnictide superconductors are close to a QTCP [19]. Neutron scattering experiments [21] have shown that lowering temperature the layered parent compound  $\text{LaFeAsO}$  first undergoes a structural phase transition, with a distortion from tetragonal structure to monoclinic structure, followed by a spin-ordering phase transition. Upon doping, magnetic order is suppressed and disappears at zero temperature (QCP). The finite-temperature magnetic transition shows basically two types of behavior [22,23]: in some compounds (particularly of the 122 family, such as  $\text{SrFe}_2\text{As}_2$ ) the transition is first order, while in others (the 1111 family, such as  $\text{LaFeAsO}$ ) it appears second order, thus suggesting that these materials are close to a tricritical point. Hence, it has been speculated [19] that opportunely tuning two non-thermal control parameter (like pressure and doping) the tricritical point can be driven to zero temperature producing a QTCP.

Theoretically the subject of classical tricritical behavior (i.e. when the tricritical point is located at finite temperature) is well known [24]. Much less has been done, instead, about quantum tricriticality, i.e. exploring the low-temperature properties and crossovers around a QTCP. The first study of quantum tricritical phase transitions was done via renormalization group (RG) theory [25], for a variety of quantum systems with and without quenched disorder, but it was carried out only at  $T = 0$ , although the general temperature-dependent flow equations were presented. Implications at finite temperature in the influence domain of a QTCP were actually missing. In the last years other few theoretical works appeared on subject related to quantum tricriticality. Schmalian and Turlakov [10] have considered the QPTs of magnetic rotons, in order to explain part of the phenomenology of the itinerant ferromagnet  $\text{MnSi}$ . Using the self-consistent Hartree approximation, they studied weak fluctuation-driven first order QPTs. For certain values of the parameters of the model under study they found a QTCP with accompanied non-Fermi liquid behavior of the electrons. In Refs.

[17,18] quantum tricriticality is analyzed by means of self-consistent renormalization theory [26] for itinerant antiferromagnets. The authors suggest that the presence of a QTCP can explain the otherwise puzzling coexistence of ferromagnetic and antiferromagnetic fluctuations observed in the heavy fermion compound  $\text{YbRh}_2\text{Si}_2$ . More recently, a functional RG analysis of quantum tricriticality in metals has been presented [27], where the correlation length has been evaluated in the quantum tricritical region, also at low-but finite temperature. Thus we clearly see that recently the relevance of quantum tricriticality has been emphasized in several theoretical and experimental works [10,11,14,16,17,18,19,27], as quantum tricritical fluctuations could give a new route to explain some of the puzzling phenomena occurring at QPTs.

In this paper, using the conventional RG approach to quantum critical phenomena [28,29], we analyze the influence at finite temperature of the QTCP and we study several crossovers occurring in the phase diagram, mainly focusing on the behavior of static susceptibility. To tackle this problem we consider the  $n$ -vector ( $O(n)$ ) generalization of the quantum action [1] for the transverse Ising model (TIM), which is one of the prototypical examples capturing the essence of QPTs, to include a variety of quantum systems which may exhibit a QTCP. We refer to these as “transverse Ising-like systems” (TIM-like systems). The TIM was originally employed to model the order-disorder transition in some double-well ferroelectric systems such as potassium dihydrogen phosphate ( $\text{KH}_2\text{PO}_4$ ) crystals [30,31,32], and since it has been successfully applied to several other materials as ferromagnets with a strong uniaxial anisotropy and further ferroelectric compounds with quantum structural phase transitions. One of the best example of experimental realization of TIM is given by the family of compounds  $\text{LiREF}_4$  (where RE stands for rare earth). In particular experimental analysis of susceptibility for the insulating magnet  $\text{LiHoF}_4$  shows that the quantum critical behavior is mean-field like [33], as expected by conventional theory on TIM. Other magnetic materials, whose QPT may be described using TIM, are some compounds belonging to the  $\text{RE}(\text{OH}_3)$  family [38]. An extensive discussion of the applications of TIM and its extensions can be found in Refs. [1,34,35,36,37]. Moreover, recent theoretical investigation on magnetic orders and lattice structure in iron-based superconductors [39,40] have shown that for high doping and low pressure the spin-density-wave transition belongs to the  $O(3)$  tridimensional universality class, with dynamical critical exponent  $z = 1$ , which enters the subject of our analysis. Hence our work may be also of some interest for these systems, where indeed the existence of a QTCP has been speculated [19].

The paper is organized as follows. In Sec.2 we present the model and the RG framework, and we derive the low temperature phase diagram. In Sec.3 we describe the quantum tricriticality and crossovers in the influence domain of a QTCP, through the analysis of the static susceptibility. Finally in Sec.4 some conclusions are drawn.

## 2 The model and the renormalization group approach

A remarkable feature in theory of critical phenomena is universality, so that through a reduced number of models one can describe a wide variety of systems. In particular for QPTs a given universality class is defined by the space dimensionality  $d$ , the order parameter symmetry index  $n$  and the dynamical critical exponent  $z$  that characterizes the intrinsic dynamics of the system. In this work we consider a  $n$ -vector Ginzburg-Landau action, which in the Fourier space is written in the form [17, 18, 25, 27] (in convenient units)

$$\begin{aligned}
 + S\{\psi\} = & \frac{1}{2} \sum_{j=1}^n \sum_{\mathbf{k}, \omega_l} (r_0 + k^2 + f(\mathbf{k}, \omega_l)) |\psi^j(q)|^2 \\
 & + \frac{u_0 T_0}{4V} \sum_{i,j=1}^n \sum_{q_1, q_2, q_3} \psi^i(q_1) \psi^i(q_2) \psi^j(q_3) \psi^j(-q_1 - q_2 - q_3) \\
 & + \frac{v_0 T_0^2}{6V^2} \sum_{i,j,l=1}^n \sum_{\{q_\nu\}} \delta_{\sum_{\nu=1}^6 q_\nu, 0} \psi^i(q_1) \psi^i(q_2) \psi^j(q_3) \\
 & \times \psi^j(q_4) \psi^l(q_5) \psi^l(q_6),
 \end{aligned} \quad (2.1)$$

where  $\psi^j(q)$  are the Fourier components of an  $n$ -vector real order parameter field,  $q \equiv (\mathbf{k}, \omega_l)$ ,  $\mathbf{k}$  is the wave vector with cutoff  $\Lambda$ ,  $\omega_l = 2\pi l T_0$  ( $l = 0, \pm 1, \pm 2, \dots$ ) are bosonic Matsubara frequencies,  $T_0$  is the temperature and  $V$  is the volume. In order to study tricritical behavior, we need to keep an expansion of the order parameter field up to sixth order. In the action (2.1) the function  $f(\mathbf{k}, \omega_l)$ , which defines the intrinsic dynamic, and the parameters  $r_0, u_0$  and  $v_0$  depend on the physical system under study. In this paper, as mentioned in the introduction, we focus on TIM-like systems for which  $f(\mathbf{k}, \omega_l) = \omega_l^2$  and the dynamical critical exponent is  $z = 1$ . However, our analysis is quite general and may be easily extended to other quantum systems with intrinsic dynamic described by  $f(\mathbf{k}, \omega_l) = |\omega_l|^\mu / k^{\mu'}$  ( $\mu \geq 1, \mu' \geq 0$ ) in the quantum action and with a dynamical critical exponent  $z = (2 + \mu')/\mu$  [41].

At mean field level (i.e. in absence of fluctuations) the considered model yields a continuous phase transition for  $u_0 > 0$  and  $r_0 = 0$ , a first-order transition for  $u_0 < 0$  and  $r_0 = 3u_0^2/16v_0$ , and a tricritical point for  $u_0 = 0$  and  $r_0 = 0$ , while the parameter  $v_0$  has to be positive for thermodynamic stability.

To analyze the critical properties of the model (2.1), taking into account the effects of fluctuations, we employ the momentum-shell RG procedure which gives a flow of Hamiltonians whose interaction parameters satisfy a set

of coupled nonlinear differential equations [25]

$$\frac{dr}{dl} = 2r + (n+2)K_d F_1(r, T)u, \quad (2.2a)$$

$$\begin{aligned} \frac{du}{dl} = & (4 - (d+z))u - (n+8)K_d F_2(r, T)u^2 + \\ & + (n+4)K_d F_1(r, T)v, \end{aligned} \quad (2.2b)$$

$$\frac{dv}{dl} = 2(3 - (d+z))v - 3(n+14)K_d F_2(r, T)uv \quad (2.2c)$$

$$\frac{dT}{dl} = zT \quad (2.2d)$$

where  $l$  is the scaling parameter,  $K_d = [2^{(d-1)}\pi^{d/2}\Gamma(d/2)]^{-1}$  and  $F_n(r, T) = T \sum_{\omega_l} [r + 1 + f(1, \omega_l)]^{-n}$ , i.e. explicitly for the systems we are analyzing with  $f(\mathbf{k}, \omega_l) = \omega_l^2$  and  $z = 1$

$$F_1(r, T) = \frac{\coth(\sqrt{1+r}/2T)}{2\sqrt{1+r}}, \quad (2.3)$$

$$F_2(r, T) = \frac{\coth(\sqrt{1+r}/2T)}{4(1+r)^{3/2}} + \frac{\text{csch}^2(\sqrt{1+r}/2T)}{8T\sqrt{1+r}} \quad (2.4)$$

Equation (2.2d) can be solved immediately yielding

$$T(l) = T_0 e^l. \quad (2.5)$$

Following the standard procedure [29], we need to separate the low and high temperature regimes, so that for the quantum regime, with  $T(l) \ll 1$  (which includes the case  $T = 0$ ), Eqs.(2.2) become

$$\frac{dr}{dl} = 2r + \frac{n+2}{2}K_d \frac{u}{\sqrt{1+r}}, \quad (2.6a)$$

$$\begin{aligned} \frac{du}{dl} = & (3-d)u - \frac{n+8}{4}K_d \frac{u^2}{(1+r)^{3/2}} + \\ & + (n+4)K_d \frac{v}{\sqrt{1+r}}, \end{aligned} \quad (2.6b)$$

$$\frac{dv}{dl} = 2(2-d)v - \frac{3}{4}(n+14)K_d \frac{uv}{(1+r)^{3/2}}, \quad (2.6c)$$

except for exponentially small corrections in  $T(l)$ . Equations (2.6) are not suitable to study quantum tricritical behavior for  $d < 2$  since, as in the corresponding classical framework [43], higher order contributions to flow equations become necessary.

At finite temperature or classical regime, with  $T(l) \gg 1$ , the recursion equations are

$$\frac{dr}{dl} = 2r + (n+2)K_d \frac{\tilde{u}}{1+r}, \quad (2.7a)$$

$$\begin{aligned} \frac{d\tilde{u}}{dl} = & (4-d)\tilde{u} - (n+8)K_d \frac{\tilde{u}^2}{(1+r)^2} + \\ & + 2(n+4)K_d \frac{\tilde{v}}{1+r}, \end{aligned} \quad (2.7b)$$

$$\frac{d\tilde{v}}{dl} = 2(3-d)\tilde{v} - 3(n+14)K_d \frac{\tilde{u}\tilde{v}}{(1+r)^2}, \quad (2.7c)$$

where the appropriate coupling parameters are  $\tilde{u} = Tu$  and  $\tilde{v} = T^2v$ .

These two regimes are separated by the condition  $T(\bar{l}) \simeq 1$  at a given value  $\bar{l}$  of the rescaling parameter  $l$ . To explore the finite temperature critical behavior we must solve the set of differential equations (2.7), using as initial conditions the solution of the set of equations (2.6) evaluated in  $\bar{l}$ . In the remaining part of the paper we will work for dimensionalities  $d \geq 3$ , where Eqs.(2.7) are suitable.

## 2.1 Solution of the flow equations

Let us first consider the quantum regime ( $T(l) \ll 1$ ). To solve the flow equations (2.6) it is convenient to introduce two new coupling parameters, which are linear combinations of the original ones,

$$t = r + \frac{n+2}{2(d-1)}K_d \left( u + \frac{n+4}{4}K_d \frac{v}{d-1} \right), \quad (2.8)$$

$$w = u + \frac{n+4}{2(d-1)}K_d v, \quad (2.9)$$

so that the equation for  $w$  (to leading order in the coupling parameters) is decoupled from the other equations. Then, the general solution of the flow equations in the low- $T$  regime, in terms of these new parameters, is given by

$$t(l) = t(0)e^{2l}Q(l)^{-(n+2)/(n+8)}, \quad (2.10a)$$

$$w(l) = w(0)e^{(3-d)l}Q(l)^{-1}, \quad (2.10b)$$

$$v(l) = v(0)e^{2(2-d)l}Q(l)^{-3(n+4)/(n+8)}, \quad (2.10c)$$

where  $Q(l) = 1 + K_d(n+8)w(0)(e^{(3-d)l} - 1)/(4(3-d))$  and

$$t(0) = r_0 + \frac{(n+2)K_d}{2(d-1)} \left( u_0 + \frac{(n+4)}{4(d-1)}K_d v_0 \right) \equiv t_0, \quad (2.11a)$$

$$w(0) = u_0 + \frac{n+4}{2(d-1)}K_d v_0 \equiv w_0, \quad (2.11b)$$

$$v(0) = v_0. \quad (2.11c)$$

The expressions (2.10) may be simplified since, working for dimensionalities  $d \gtrsim 3$ , we have  $Q(l) \simeq 1$  for  $l \gg 1$ . This is equivalent to linearizing the RG equations around the gaussian fixed point (GFP), which is the one that governs tricriticality. Basically, we have that fluctuations shift the conditions for tricriticality given by Landau theory from  $(r_0 = 0, u_0 = 0)$  to  $(t_0 = 0, w_0 = 0)$ . Hence we have continuous phase transitions for  $t_0 = 0$  and  $w_0 > 0$  (or  $u_0 > -(n+4)K_d v_0/2(d-1)$  so that also negative values of  $u_0$  are allowed) and a tricritical point for  $t_0 = 0$  and  $w_0 = 0$ . For  $w_0 < 0$  there are no stable fixed points, so nothing can be said in this region within the RG approach, but matching with Landau theory we can assume that first order phase transitions occur. Restoring the original parameters at  $T_0 = 0$  (from the conditions  $t_0 = w_0 = 0$ ) we obtain the coordinates of the QTCP

$$r_{tr} = \frac{(n+2)(n+4)K_d^2 v_0}{8(d-1)^2}, \quad u_{tr} = -\frac{(n+4)K_d v_0}{2(d-1)}. \quad (2.12)$$

We stress that fluctuations shift the coordinate  $u_0 = 0$  of the tricritical point, as predicted from Landau theory, to

the lower value  $u_0 = u_{tr} (< 0)$ . We also determine the line of QCPs in the  $(r_0, u_0)$ -plane

$$r_c = -\frac{n+2}{2(d-1)}K_d \left( u_0 + \frac{n+4}{4(d-1)}K_d v_0 \right), \quad (2.13)$$

parameterized by  $v_0 > 0$ . This equation defines a straight line which ends at the QTCP with  $r_c$  increasing from negative to positive values up to the QTCP coordinate  $r_c = r_{tr} > 0$ , by variation of the parameter  $u_0$  from positive to negative values provided that  $w_0 \geq 0$ .

We now provide an estimate of the line separating the classical and quantum regime, i.e. when  $T(\bar{l}) \simeq 1$ , focusing on the scaling field  $t(l)$  which contains the relevant information about the structure of the phase diagram and the physics of the systems. We know that one can stop the scaling when  $t(l) \sim 1$  [29,44], hence with the approximate solution  $t(l) \simeq t_0 e^{2l}$  close to the GFP, we obtain for the rescaling parameter a value  $l^* \gg 1$  ( $e^{l^*} = t_0^{-1/2}$ ) which can be used to evaluate the renormalized temperature  $T(l)$ , requiring that  $T(l^*) \ll 1$  in order to stay within the quantum regime. This gives us the condition  $T_0 \ll t_0^{1/2} = (r_0 - r_c)^{1/2}$  for the occurrence of this asymptotic regime. In particular, if  $u_0 = u_{tr}$  we have  $T_0 \ll (r_0 - r_{tr})^{1/2}$ . Hence there is a crossover line, separating quantum ( $T(l) \ll 1$ ) and classical ( $T(l) \gg 1$ ) regimes for  $r_0 > r_{tr}$ , which is given by

$$T^\dagger \simeq (r_0 - r_{tr})^{1/2}. \quad (2.14)$$

As mentioned before, to study the low-temperature phase diagram we have to solve the set of coupled differential equations (2.7) appropriate at finite temperature, where we assume as initial conditions the solution (2.10) of the quantum regime evaluated at  $\bar{l} = \ln(1/T_0)$ .

Also in the classical regime it is convenient to rewrite the recursion relations by introducing two new coupling parameters

$$\tilde{t} = r + \frac{n+2}{d-2}K_d \left( \tilde{u} + \frac{n+4}{d-2}K_d \tilde{v} \right), \quad (2.15)$$

$$\tilde{w} = \tilde{u} + \frac{2(n+4)}{d-2}K_d \tilde{v}, \quad (2.16)$$

in terms of which the solution of Eqs.(2.7) is

$$\tilde{t}(l) = \tilde{t}(\bar{l})e^{2(l-\bar{l})}\tilde{Q}(l, \bar{l})^{-(n+2)/(n+8)}, \quad (2.17a)$$

$$\tilde{w}(l) = \tilde{w}(\bar{l})e^{(4-d)(l-\bar{l})}\tilde{Q}(l, \bar{l})^{-1}, \quad (2.17b)$$

$$\tilde{v}(l) = \tilde{v}(\bar{l})e^{2(3-d)(l-\bar{l})}\tilde{Q}(l, \bar{l})^{-3(n+4)/(n+8)}, \quad (2.17c)$$

where  $\tilde{Q}(l, \bar{l}) = 1 + (n+8)K_d \tilde{w}(\bar{l})(e^{(4-d)(l-\bar{l})} - 1)/(4-d)$ . The results in the classical regime are very similar to those obtained in Ref. [42]. For  $3 \lesssim d < 4$  there is a GFP which is doubly unstable having two relevant scaling fields,  $\tilde{t}(l)$  and  $\tilde{w}(l)$ , and a critical FP which is unstable only against the scaling field  $\tilde{t}(l)$ . Thus, when  $\tilde{t}(\bar{l}) = 0$  with  $\tilde{w}(\bar{l}) > 0$ , the Hamiltonian flow evolves towards the critical FP, yielding a continuous phase transition, while for  $\tilde{w}(\bar{l}) < 0$  the critical FP is no longer accessible. Therefore, the conditions

$\tilde{t}(\bar{l}) = 0$  and  $\tilde{w}(\bar{l}) = 0$  define a finite-temperature tricritical point, marking the borderline between these two regimes. The dependence on  $\bar{l}$  implies a dependence on temperature  $T_0$  (from now on we will omit the subscript 0 for the physical temperature and for the other coupling parameters, too). Hence the condition  $\tilde{t}(\bar{l}) = 0$  with  $\tilde{w}(\bar{l}) > 0$  determines the equation for the critical surface at finite temperature, while  $\tilde{t}(\bar{l}) = 0$  with  $\tilde{w}(\bar{l}) = 0$  provides the finite-temperature tricritical line.

## 2.2 Critical and tricritical lines and low temperature phase diagram

Using the solution of the RG equations derived in previous subsection and the conditions for the relevant scaling fields we obtain the second-order phase transition surface in the parameters space

$$r_c(T) = r_c - a_{n,d}(u - u_{tr})T^{d-1} - b_{n,d}vT^{2(d-1)}, \quad (2.18)$$

under the condition for  $u$

$$\begin{aligned} u &> -\frac{n+4}{2(d-1)}K_d v \left(1 + \frac{3d-2}{d-2}T^{d-1}\right) \\ &= u_{tr} \left(1 + \frac{3d-2}{d-2}T^{d-1}\right), \end{aligned} \quad (2.19)$$

where  $r_c$  (which explicitly depends on  $u$  and  $v$ ) is given by Eq.(2.13) and

$$a_{n,d} = \frac{d(n+2)K_d}{2(d-1)(d-2)}, \quad (2.20)$$

$$b_{n,d} = \frac{5d^2 - 8d + 4}{8(d-1)^2(d-2)^2}(n+2)(n+4)K_d^2. \quad (2.21)$$

For fixed values of  $u > u_{tr}$  and for  $v > 0$ , we obtain a family of critical lines in the  $(r, T)$ -plane, ending in QCPs. At sufficiently low temperature close to the QCP

$$r_c(T) = r_c - a_{n,d}(u - u_{tr})T^\psi, \quad (2.22)$$

where  $\psi = d - 1$ . In this way we recover the usual shift exponent for systems in the universality class which is the subject of this paper. Increasing the temperature, a different  $T$ -behavior with exponent  $2\psi$  governs  $r_c(T)$ . Then, a crossover temperature  $T^*$  exists separating two distinct asymptotic  $T$ -behaviors of the phase boundary, with power-laws  $\psi$  for  $T \ll T^*$  and  $2\psi$  for  $T \gg T^*$ , with

$$T^* = \left[ \frac{a_{n,d}(u - u_{tr})}{b_{n,d}v} \right]^{1/\psi}. \quad (2.23)$$

When  $u = u_{tr}$ , among the family of critical lines in the  $(r, T)$ -plane, parameterized by  $u$  and  $v$ , we select the one ending in the QTCP, characterized by the temperature behavior

$$r_{qtc}(T) = r_{tr} - b_{n,d}vT^{2\psi}, \quad (2.24)$$

where the index  $qtc$  stands for “quantum tricritical”.

In terms of temperature, this peculiar phase boundary is determined by the equation

$$T_{qtc}(r) = \left( \frac{r_{tr} - r}{b_{n,d}v} \right)^{1/2\psi} \quad (\text{for } r < r_{tr}), \quad (2.25)$$

which provides the shift exponent  $\phi = 1/2\psi = 1/2(d-1)$ , to be compared with  $\phi = 1/\psi = 1/(d-1)$  which occurs when the critical lines ends in an ordinary QCP. Of course as  $d \rightarrow 3^+$ , we should have  $\phi = 1/2$  close to a QCP for  $T \ll T^*$  and  $\phi = 1/4$  close to the QTCP.

From the conditions  $\tilde{t}(\bar{l}) = 0$  and  $\tilde{w}(\bar{l}) = 0$ , as written above, we get the expression of the finite-temperature tricritical line (TCL)

$$r_{tcl}(T) = r_{tr} \left[ 1 + \frac{2(3d-2)}{d-2}T^\psi + O(T^{2\psi}) \right]. \quad (2.26)$$

If we consider the projection of the TCL in the  $(r, T)$ -plane we obtain a tricritical shift exponent given by  $\phi_{tr} = 1/\psi = 1/(d-1)$ , which is the same value we obtain for conventional criticality. In particular for  $d \simeq 3$  we have  $\phi_{tr} = 1/2$ ,

In Fig. 1 we present the schematic low- $T$  phase diagram in the  $(r, u, T)$ -space, where we draw the critical surface which is given by Eq. (2.18), provided that  $\tilde{w}(\bar{l}) > 0$  (see Eq. (2.19)). We also draw the surface  $\tilde{w}(\bar{l}) = 0$ , behind which there is the region of the phase diagram ( $\tilde{w}(\bar{l}) < 0$ ) that is inaccessible within our RG treatment and where first-order phase transitions are expected to occur. The intersection between these two surfaces gives the TCL at finite temperature  $r_{tcl}(T)$ , given by Eq. (2.26).

In the next section we are going to explore the critical behavior and crossovers around the peculiar second-order phase transition line  $r_{qtc}(T)$  which ends in the QTCP. In this way we will see how the finite-temperature behavior of observable quantities modifies due to the influence of the QTCP as a special QCP.

## 3 Criticality and crossovers in the influence domain of the quantum tricritical point

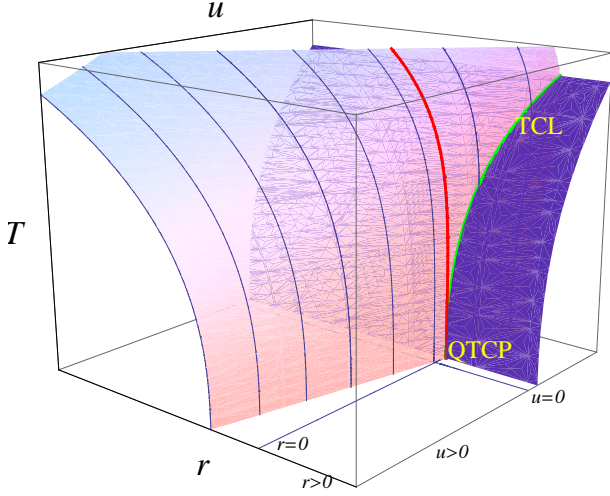
According to the conventional RG framework [44,45], the correlation length  $\xi$  and the static susceptibility  $\chi \propto \xi^{2-\eta}$  can be expressed as

$$\xi = \xi_0 e^{l^*}, \quad \chi = \chi_0 e^{(2-\eta)l^*}, \quad (3.1)$$

where  $\eta$  is the Fisher exponent and the scale  $l^* \gg 1$  is determined setting  $\tilde{t}(l^*) \simeq 1$  [44]. Here, the constants  $\xi_0$  and  $\chi_0$  are inessential for our purposes, and will be included in the definition of  $\xi$  and  $\chi$  in the next developments. In our analysis the Fisher exponent is  $\eta = 0$ .

Here, we focus on the behavior of susceptibility  $\chi$ . From  $\tilde{t}(l^*) \simeq 1$  and Eq. (2.17a) we obtain the self consistent equation for the static susceptibility

$$\chi^{-1} \left[ 1 + K_d \frac{n+8}{4-d} g_{n,d}(T, u) T \chi^{2-\frac{d}{2}} \right]^{\frac{(n+2)}{(n+8)}} = r - r_c(T), \quad (3.2)$$



**Fig. 1.** Schematic phase diagram in the  $(r, u, T)$ -space for  $n \geq 1$  and  $3 \lesssim d < 4$ . The critical surface is the light one and the lines on it are critical lines ending in ordinary QCPs for different fixed values of  $u > u_{tr}$ , while the thicker line (red online) is the one ending in the QTCP, named  $r_{qtc}(T)$  in the text. The dark surface in the back is given by  $\tilde{w}(\tilde{l}) = 0$  (see text). In the region of the phase diagram behind this surface first-order phase transitions are expected to occur. The intersection of the two surfaces (green online) is the finite-temperature tricritical line  $r_{tcl}(T)$ .

where

$$g_{n,d}(T, u) = u - u_{tr} \left( 1 + \frac{3d-2}{d-2} T^{d-1} \right). \quad (3.3)$$

We consider the peculiar critical line which ends in the QTCP, thus we fix the value of the parameter  $u$  to the coordinate  $u_{tr}$  of the QTCP.

The self-consistent equation (3.2) for  $\chi$  then becomes

$$\chi^{-1} \left[ 1 + c_{n,d} v T^d \chi^{2-\frac{d}{2}} \right]^{\frac{(n+2)}{(n+8)}} = r - r_{qtc}(T), \quad (3.4)$$

where  $c_{n,d} = \frac{(n+8)(n+4)(3d-2)K_d^2}{2(d-1)(d-2)(4-d)}$ .

If we consider isothermal paths in the  $(r, T)$ -phase diagram, we obtain two asymptotic regimes for the static susceptibility, which correspond to mean-field behavior and “classical Wilsonian” one, namely

$$\chi \simeq (r - r_{qtc}(T))^{-\gamma}, \quad (3.5)$$

with

$$\gamma = \begin{cases} 1, & \text{for } r \gg r_G^{tr}(T) \\ 1 + \frac{n+2}{2(n+8)}(4-d), & \text{for } r_{qtc}(T) < r \ll r_G^{tr}(T) \end{cases} \quad (3.6)$$

where

$$r_G^{tr}(T) \simeq r_{qtc}(T) + (c_{n,d} v)^{2/(4-d)} T^{2d/(4-d)} \quad (3.7)$$

is the crossover line which separates these two regimes, which we call Ginzburg line since it is related to the Ginzburg criterion. To obtain the expected expression (3.6) for  $\gamma$  in the critical or Wilsonian (W) region, we have used an expansion to first order in  $4-d$ , as it is normally done for classical continuous phase transitions. Thus we correctly recover the critical behavior close to a finite-temperature critical point [46, 47]. For conventional QCP, the corresponding Ginzburg line, associated to the generic critical line Eq.(2.18), is given by (see Eq.(3.2))

$$r_G(T) = r_c(T) + \left\{ K_d \frac{n+8}{4-d} \left[ u + \frac{n+4}{2(d-1)} K_d v \left( 1 + \frac{3d-2}{d-2} T^{d-1} \right) \right] T \right\}^{2/(4-d)}. \quad (3.8)$$

If we approach the critical line fixing the value of the non-thermal parameter  $r$  (with  $r < r_{tr}$ ), decreasing temperature, Eq. (3.4) can be rewritten as

$$\chi^{-1} \left[ 1 + c_{n,d} v T^d \chi^{2-d/2} \right]^{(n+2)/(n+8)} \simeq 2b_{n,d} \psi T_c(r)^{2\psi-1} (T - T_c(r)), \quad (3.9)$$

where we have used, in the right side of Eq.(3.4), an expansion to first order in  $T - T_c(r)$  (i.e. for temperatures close to the critical one for a given value of  $r$ ). Then we recover again the classical exponent  $\gamma$ , which is the same as for isothermal paths (see eqs. (3.6)) to first order in  $4-d$ .

The most experimentally interesting situation occurs when we fix both  $u$  and  $r$  at the QTCP values (Eq. (2.12)) and we study the behavior of susceptibility decreasing temperature. We call this line, with  $r = r_{tr}$ , “quantum tricritical trajectory” in analogy with the usual quantum critical case [46]. The self-consistent equation (3.4) then becomes

$$\chi^{-1} \left( 1 + c_{n,d} v T^d \chi^{2-\frac{d}{2}} \right)^{\frac{(n+2)}{(n+8)}} = b_{n,d} v T^{2\psi}. \quad (3.10)$$

At sufficiently low temperature we obtain

$$\chi \simeq (b_{n,d} v)^{-1} T^{-2\psi} \sim T^{-2(d-1)}, \quad (3.11)$$

which differs from the behavior occurring close to a conventional QCP ( $\chi \sim T^{-\psi}$ ) by the factor 2 in the exponent. Thus we still have a power-law behavior in temperature when we are in the influence domain of a QTCP, but with a different power compared to the case of a QCP.

Let us now consider the region of the phase diagram for values  $r > r_{tr}$ , where the self-consistent equation for  $\chi$  is properly written as

$$\chi^{-1} \left[ 1 + c_{n,d} v T^d \chi^{2-\frac{d}{2}} \right]^{\frac{(n+2)}{(n+8)}} = r - r_{tr} + b_{n,d} v T^{2\psi}. \quad (3.12)$$

We distinguish the cases in which  $(r - r_{tr})$  dominates over the  $T$ -dependent term on the right side of (3.12), and

viceversa. So, at sufficiently high temperature, i.e. for  $T \gg T^{\dagger\dagger}(r)$ , where

$$T^{\dagger\dagger}(r) = \left( \frac{r - r_{tr}}{b_{n,d}v} \right)^{1/2\psi}, \quad (3.13)$$

the behavior of susceptibility (and hence other observable quantities) is essentially the same as the one occurring along the quantum tricritical trajectory.

Below this crossover line (i.e. for  $T < T^{\dagger\dagger}(r)$ ) we have to be careful to the presence of the other crossover line (2.14), which separates the classical and quantum regime. Indeed, the self-consistent equation obtained for  $\chi$  in this section, coming from the solutions of the RG recursion relations (2.7) at finite temperature, is not valid in the quantum regime (i.e. for  $T < T^{\dagger}(r)$ , which we call  $Q_1$  regime). Hence for  $T^{\dagger}(r) < T < T^{\dagger\dagger}(r)$ , we have

$$\chi^{-1} \simeq (r - r_{tr}) + b_{n,d}vT^{2\psi}, \quad (3.14)$$

where now  $r - r_{tr}$  dominates and  $b_{n,d}vT^{2\psi}$  represents the leading  $T$ -dependent deviation from the ( $T = 0$ ) mean-field behavior occurring in the quantum regime. Hence, even if we are above  $T^{\dagger}(r)$ , we call  $Q_2$  this region of temperature.

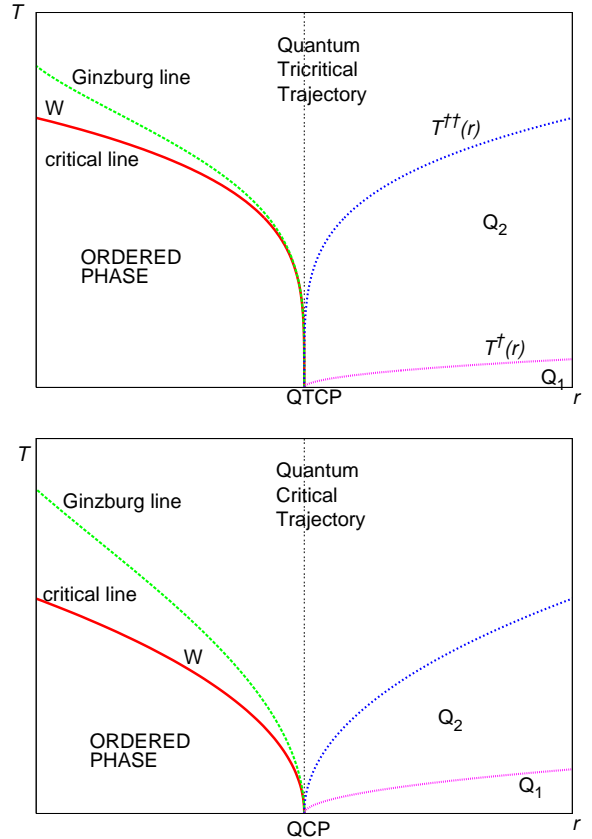
In the  $Q_1$  regime the present formulation misses the exponentially small correction in temperature (see Ref. [46]) to the leading mean field behavior of susceptibility  $\chi \sim (r - r_{tr})^{-1}$ .

In Fig. 2 we present the low-temperature phase diagram in the  $(r, T)$ -plane in the influence domain of a QTCP emerging from our analysis and, for comparison, also the corresponding phase diagram that we obtain when the critical line ends in an usual QCP.

This figure suggests that the phase diagrams in the  $(r, T)$ -plane close to a QCP and a QTCP (as a special QCP) appear very similar and hence hard to distinguish in experiments at very low temperatures. However an observed deviation from the conventional quantum criticality could be simply explained as a manifestation of the different nature of fluctuations in the influence domains of QCP and QTCP. Besides, it clearly emerges that, close to the QTCP, the fluctuations are sensibly weaker than those in the influence region of a conventional QCP. From Eqs. (3.7) and (3.8), we have indeed that the amplitudes of the W-regions are given by  $\Delta_G^{tr}(T) = r_G^{tr}(T) - r_{qtc}(T) \sim T^{2d/(4-d)}$  and  $\Delta_G(T) = r_G(T) - r_c(T) \sim T^{2/(4-d)}$  (for  $u > 0$ ), so that one has always  $\Delta_G^{tr} < \Delta_G(T)$  for  $3 \leq d < 4$ . This leads to the experimentally interesting conclusion that the quantum tricriticality is well described essentially in terms of mean field exponents, combined with the shift exponent  $\phi = 1/2\psi = 1/2(d-1)$ , also very close to the full critical line which ends at the QTCP.

## 4 Conclusions

In the present paper we have explored the quantum tricriticality, i.e. the low-temperature properties and crossovers



**Fig. 2.** Low-temperature phase diagram in the  $(r, T)$ -plane in the presence of a QTCP (top) and when the critical line ends in a QCP (bottom). The nomenclature of the regions is the same for both figures: above the critical line we have the Wilsonian (W) critical region where we recover classical critical exponents; on the right side of the phase diagrams we have the “quantum” regimes  $Q_1$  and  $Q_2$ , where the behavior of the susceptibility is essentially mean-field in terms of  $(r - r_{tr})$  or  $(r - r_c)$  with small corrections in temperature. The main differences occur in the fan-shaped region above the QTCP and the QCP, where the static susceptibility behaves as  $\chi \sim T^{-2\psi}$  and  $\chi \sim T^{-\psi}$  respectively. Notice also the different shape of the phase boundaries and the size of the Wilsonian regions between the two cases.

around the QTCP, of a variety of  $d$ -dimensional systems with transverse Ising-like symmetry which could exhibit a QTCP by variation of the effective coupling parameters in the quantum Ginzburg-Landau-Wilson action. The study has been performed via a perturbative RG approach in the Hertz-Millis spirit, which works very well for dimensionalities  $3 \leq d < 4$ , focusing on the shape of the phase boundaries and on the behavior and crossovers of susceptibility or correlation length. It is worth emphasizing that, in this range of dimensionalities, where tricriticality is governed by a Gaussian fixed point (GFP) and criticality by the critical FP, the coupling parameter  $v$  (or  $\tilde{v}$  in the classical regime), which enters the problem through the  $O(\psi^6)$  term in the quantum action (2.1), flows towards zero under iteration of the RG transformation and hence it is an



“irrelevant” variable (in the RG sense). Nevertheless, it plays a crucial role in determining the correct physics and it must be necessarily included in the RG analysis since the initial condition  $v > 0$  is essential for thermodynamic stability whenever  $u < 0$  [42]. Of course the  $O(\psi^6)$  coupling becomes marginal at  $d = 3$  in the classical regime implying logarithmic corrections to the Gaussian tricriticality. We note also that in our RG scenario, as an effect of fluctuations, the key parameters which govern criticality and tricriticality (and the related crossovers) are not the original ones  $r$  and  $u$  but rather the new scaling fields  $t$  and  $w$  (here for formal simplicity we refer to  $t$  and  $w$  both for classical or quantum regime), with  $w > 0$  implying second-order phase transitions and  $w < 0$  first-order ones. So, tricriticality is approached when  $t$  and  $w$  go to zero with  $v > 0$ , by variation of temperature and the original coupling parameters. In particular, the tricritical line, ending in the QTCP, is determined by the equations  $t = w = 0$  (and not by  $r = u = 0$  as in the Landau theory). Remarkably, the inclusion of fluctuations produces also a shift of the phase boundaries in the global phase diagram with respect to that resulting from the Landau theory (see. Fig. 1). The emergent quantum tricritical scenario, as compared with that induced by QCP-fluctuations, provides some peculiar features which could be very useful in interpreting eventual deviations from the predictions expected in the influence domain of a conventional QCP. In particular, we find three key ingredients which may be of experimental interest. The first one consists in a peculiar phase diagram where the amplitude of the Wilsonian classical critical region around the selected critical line ending in the QTCP is considerably smaller than that occurring near the standard quantum critical regime. Next, a tricritical shift exponent  $\phi^{tr}$  is derived which is identical to that found in the conventional quantum criticality. Finally, a peculiar crossover of the critical shift exponents (which characterize the low-temperature behaviors of different critical lines) from the conventional value  $\phi = 1/(d - 1)$  to the new one  $\phi = 1/2(d - 1)$  takes place when the system under study is tuned between quantum criticality and quantum tricriticality by variation of the non-thermal control parameter. It is worth emphasizing again that the QTCP scenario emerging from our analysis has been obtained for the particular universality class of  $n$ -vector quantum systems with TIM-like dynamics. However, it can be easily extended to include other quantum materials (as, for instance, the oxides and the heavy-fermion compounds mentioned in the introduction) where one expects that a QTCP may be induced by variations of certain external parameters. For a class of metals, a study of the crossovers between quantum criticality and quantum tricriticality has been recently performed [27] by using a functional RG approach. However, it invokes the Hertz action and involves numerous approximations, as the linearization of the flow equations with the aim to obtain an analytical solution, which are not easily controllable, especially in describing crossover phenomena. This is avoided in our analysis and our predictions support the use of a direct perturbative RG framework for obtaining

reliable analytical results, at least around realistic dimensionalities. In this sense, it constitutes again a good tool to have a suitable, although qualitative, scenario concerning the essential physics which occurs around a conventional QCP and around a QTCP and the crossovers between these two asymptotic regimes as a manifestation of the interplay between quantum critical and tricritical fluctuations.

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